Recap 18.032

Tristan Ozuch

1 Solving first order ODEs

Sometimes we are lucky enough to have equations that we can solve. In these situation, we should almost always do it and analyze the obtained solutions.

1.1 Integration

The unique solution to

$$\begin{cases} y' = f(t) \\ y(t_0) = y_0 \end{cases}$$

is $y(t) = y_0 + \int_{t_0}^t f$.

The set of solutions of $y' = f(t)$ is

$$\{y(t) = c + \int_{t_0}^t f, c \in \mathbb{R}\}.$$

1.2 Separable equations

If you reduce your equation to the form

$$y'(t)g(y(t)) = f(t)$$

integrate both sides to find

$$\int_{y(t_0)}^{y(t)} g = \int_{t_0}^t f + cst.$$

Then try and solve in $y(t)$ and verify that it gives you a solution.

1.3 First order linear ODE

The unique solution to

$$\begin{cases} y' + p(t)y = f(t) \\ y(t_0) = y_0 \end{cases}$$

is $y(t) = y_0 \exp\left(-\int_{t_0}^t p\right) + \exp\left(-\int_{t_0}^t p\right) \int_{t_0}^t \exp\left(\int_{t_0}^s p\right) f(s)ds$.

The set of solutions of $y' + p(t)y = f(t)$ is

$$\{y(t) = c\exp\left(-\int_{t_0}^t p\right) + \exp\left(-\int_{t_0}^t p\right) \int_{t_0}^t \exp\left(-\int_{t_0}^s p\right) f(s)ds, c \in \mathbb{R}\}.$$

2 Existence-Uniqueness theorem

In most situations, we cannot solve our ODEs explicitly. We can only show existence and uniqueness of solutions which is a powerful tool to analyze qualitatively solutions of ODEs.

2.1 Local version

Theorem 2.1. *Assume that on a small region of interest around* (t_0, y_0)*:*

1. $t \mapsto f(t, y)$ *is continuous and*
2. $y \mapsto f(t, y)$ *is L-Lipschitz.*

Then, ***there is*** *a* ***unique*** *solution to*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

on $[t_0 - c, t_0 + c]$.

Without the existence part of this theorem, the study of ODEs would not even make sense. You can then start your proof by saying 'let y be a solution of the ODE...', now that you know it exists. More concretely, the uniqueness part lets you prove that two solutions equal at **one** point are equal at **all** points! It also lets you show symmetries of solutions by setting up the "right other solution" from a first one with the same IVP. It finally lets you prove inequalities between solutions by contradiction.

Several steps of proof are extremely important for later:

- Iteration of the contracting operator $F : C^1 \to C^1$

$$F : y \mapsto \left(t \mapsto y_0 + \int_{t_0}^{t} f(s, y(s)) ds \right)$$

  to find a fixed point: $y = F(y)$.
- Pointwise convergence,
- Uniform convergence,
- **Theorem:** A uniform limit of continuous functions is continuous,
- Uniform convergence requires quantitative estimates.
- Cauchy sequences and the completeness of $\mathbb{R}$.

We also discussed the continuous dependence on parameters/initial condition. We obtained an estimate on the difference between solutions thanks to:

Theorem 2.2 (Grönwall's Lemma)**.** *Assuming that everything is positive and* $t > t_0$*. If*

$$z(t) \leqslant C + \int_{t_0}^{t} p(s) z(s) ds,$$

then

$$z(t) \leqslant C \exp\left(\int_{t_0}^{t} p \right).$$

2.2 Global existence-uniqueness

Theorem 2.3. *If $f : I \times \mathbb{R}$ is continuous and locally Lipschitz with respect to the second variable, then for all $(t_0, y_0) \in I \times \mathbb{R}$ there exists a unique solution of*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

has a unique ***maximal interval of existence*** *included in I.*

Moreover if one of the ends of this maximal interval of existence is not an end of I, then the solution y cannot stay bounded when approaching this point.

The scheme of proof is also important for later:

- there is a **connectedness argument**.
- If a sub interval of I is both open and closed, then it is either empty or I.
- The closeness of $\{t \mid y(t) = z(t)\}$ is often a consequence of the continuity of y and z.
- The openness rather comes from local uniqueness.

If you know that f is bounded, this gives you global existence-uniqueness:

Corollary 2.4. *If f is additionally bounded by a common value on I, then the maximal interval of existence is I.*

If you know that f is **uniformly** Lipschitz, this gives you global existence-uniqueness:

Corollary 2.5. *If f is L-Lipschitz with respect to the second variable, then the maximal interval of existence is I.*

3 Linear second order ODEs

3.1 Some linear algebra

We mainly saw definitions: vector (sub)space, linear operator, Kernel, Image, affine subspace.

The set of solutions of a **linear homogeneous** ODE is a vector subspace. The set of solutions of a linear ODE is an affine subset.

We also have discussed the notion of dimension through linearly independent families, spanning families and bases. The largest linearly independent families are the smallest spanning families and are bases. Their common number of vectors is the dimension.

We have also seen ways of bounding the dimension of spaces and applied it to proving that the set of solutions to a linear second order ODE is of dimension at most 2.

3.2 Linear second order ODEs with constant coefficients

Second order ODEs with constant coefficients are among the rare equations that we can explicitly solve.

Consider the equation

$$y'' + py' + qy = 0$$

for **constants** $p, q \in \mathbb{R}$.

The **characteristic polynomial** of this equation is

$$P : s \mapsto s^2 + ps + q.$$

Theorem 3.1. *The set of solutions of*

$$y'' + py' + qy = 0$$

for ***constants*** $p, q \in \mathbb{R}$ *is given by:*

- *If* $\Delta = p^2 - 4q > 0$*, then there are two* ***distinct real*** *roots* α *and* β *and the set of solutions is:*

$$\{t \mapsto Ae^{\alpha t} + Be^{\beta t} | A, B \in \mathbb{R}\},$$

- *If* $\Delta = p^2 - 4q = 0$*, then there is one* ***real double*** *root* α *and the set of solutions is:*

$$\{t \mapsto Ae^{\alpha t} + Bte^{\alpha t} | A, B \in \mathbb{R}\}, \text{ and}$$

- *If* $\Delta = p^2 - 4q < 0$*, then there are two complex* ***conjugate solutions*** $a \pm ib$ *and the set of solutions is:*

$$\{t \mapsto e^{at}(A\cos(bt) + B\sin(bt)) | A, B \in \mathbb{R}\}.$$

3.3 Wronskian and finding new solutions

The **Wronskian** of two differentiable functions f and g is

$$W(f, g) := fg' - gf'.$$

It is an antisymmetric bilinear form. The first link with linear second order ODEs is the following result.

Theorem 3.2. *Let* f *and* g *be a solution of*

$$y'' + p(t)y' + q(t)y = 0.$$

Then for all t

$$W(f, g)(t) = W(f, g)(t_0) \exp\left(-\int_{t_0}^{t} p\right),$$

meaning that up to a real constant $W(f, g)(t_0)$*, the value does not depend on* f *or* g*.*

Theorem 3.3. *We have equivalence between:*

1. *for all* t*,* $W(f, g)(t) \neq 0$*,*
2. (f, g) *is a linearly independent family,*
3. *there exists* t_0 *so that* $W(f, g)(t_0) \neq 0$*.*

It is however much easier to show that two functions are linearly independent by showing that they are not proportional...

We mostly use the Wronskian to find new solutions.

Theorem 3.4. *Let* f *be a solution of*

$$y'' + p(t)y' + q(t)y = 0$$

which does not vanish on an interval J*.*

Then, for all $t_0 \in J$ *the function defined by*

$$g(t) := f(t) \int_{t_0}^{t} \frac{\exp\left(-\int_{t_0}^{s} p\right)}{f^2(s)} ds$$

is a solution of $y'' + p(t)y' + q(t)y = 0$ *and* f *and* g *are linearly independent.*

This essentially means that whenever we know **one** solution to a second order linear ODE, we are able to recover the whole set of solutions.

3.4 Solving the non homogeneous problem: Variation of parameters

The trick to solve nonhomogeneous ODEs is to consider our set of solutions and replace the *parameters* of the solutions of the **homogeneous** equation by *functions* to solve the **non homogeneous** (or simply linear) problem.

The method is important – go back to the notes. The end result is that if (y_1, y_2) is a basis of solutions of the **homogeneous** linear problem

$$y'' + p(t)y' + q(t)y = 0,$$

that is, any solution may be written as $t \mapsto c_1 y_1(t) + c_2 y_2(t)$ for *constant numbers* c_1 and c_2.

One solution to the non homogeneous problem

$$y'' + p(t)y' + q(t)y = f(t),$$

can be found of the form $t \mapsto c_1(t)y_1(t) + c_2(t)y_2(t)$ where this time $t \mapsto c_i(t)$ are *functions*. They are solutions of:

$$c_1'(t) = -\frac{y_2(t)f(t)}{W(y_1, y_2)(t)}$$

$$c_1'(t) = \frac{y_1(t)f(t)}{W(y_1, y_2)(t)}.$$

This technique will extend to **any** order.

4 Common themes in analysis of PDEs

The goal here is to give you an overview in a simpler setting of crucial ideas in analysis which are often first encountered in very abstract settings where the core concepts can be hidden by other difficulties.

4.1 Boundary value problems

Instead of conditions on the value of a function and its derivatives at a given point, we can impose conditions at different points.

These usually do not have as nice properties of existence/uniqueness however. Through maximum principle, you still showed the uniqueness of solutions to second order ODEs

$$y'' + p(t)y' + q(t)y = f(t)$$

assuming that $q < 0$.

4.2 Maximum principle

Here the principle is simple: if a C^2 function u has a local maximum at an interior point x_0 of an interval, then, it satisfies:

1. $u'(x_0) = 0$, and
2. $u''(x_0) \leqslant 0$.

When u moreover satisfies an ODE, then, this can be used to prove that u cannot have a maximum of a given sign by arguing by contradiction.

Theorem 4.1. *Consider*

$$y'' + p(t)y' + q(t)y = f(t)$$

where $q(t) < 0$ and $f(t) \geqslant 0$ on (a,b). Then, there cannot be a positive ***local*** *maximum of y on (a,b).*

If for all $t \in (a,b)$, $y'' + p(t)y' > 0$ (in particular if $q = 0$), then for all $t \in (a,b)$, we have $y(t) \leqslant \max(y(a), y(b))$.

The idea to keep in mind is that interior extrema of a function satisfying an ODE (or PDE) impose conditions on the sign of the coefficients.

4.3 Weak solution of ODEs

This is more of a chapter for your culture, it will soon be central in your study of PDEs: most articles in theoretical and applied mathematics use Sobolev (or Hölder) spaces. Remember the definitions of L^2, H^k and relate them to functions having Fourier decompositions.

The idea to keep in mind is that through **integration by parts** against a class of *test functions*, one can define weak derivatives of functions, hence weak solutions of ODEs (or PDEs).

We also discussed how one can make sense of the linearization of a *functional*, that is a function taking function in a given class of regularity.

There are some ideas and formulas to remember:

Cauchy-Schwarz inequality: if $f, g : (a,b) \to \mathbb{R}$ are L^2, then:

$$\Big(\int_a^b f(t)g(t)dt\Big)^2 \leqslant \int_a^b f(t)^2 dt \int_a^b g(t)^2 dt.$$

You can therefore control the integral of a product by the products of the L^2-norms.

A useful inequality: if $a, b \in \mathbb{R}$, then, for all $c > 0$, one has:

$$2ab \leqslant ca^2 + c^{-1}b^2.$$

We use it to control the product of two numbers or functions: it *decouples* a and b. The parameter c can be chosen to yield better estimates, e.g. when $a^2 \ll b^2$, one would choose $c = b/a$.

Good exercise: Once we have dealt with Fourier series, you can prove that a H^k-weak solution has a Fourier decomposition $\sum_{n\in\mathbb{Z}} n^{2k}|c_n|^2 < \infty$. For this, you can take a smooth test function ϕ (that is with fast decaying Fourier series – and in particular sines, cosines and complex exponentials themselves) and compute the integrals and integrate by parts.

5 (pre-)Hilbert spaces geometry

Here again definitions are the most important for now. You need to know what a *scalar (or inner) product* is and how it is used to define *orthogonality* and *symmetric* operators.

The definition of *eigenvalue* is also central. It has been used and will be used even more towards the end of the class as follows.

Theorem 5.1. *In a vector space E, a family of vectors which are eigenvectors (or eigenfunctions) of a given linear operator $L : E \to E$ associated to* ***disctinct*** *eigenvalues is linearly independent.*

In the case of symmetric operators, we have a stronger result.

Theorem 5.2. *In a vector space equipped with a scalar product $(E, \langle , \rangle)$, a family of vectors which are eigenvectors (or eigenfunctions) of a given **symmetric** linear operator $L : E \to E$ associated to **disctinct** eigenvalues is orthogonal.*

Another central formula once one has an **orthonormal** basis $(e_i)_i$ of a vector space equipped with a scalar product $\langle , \rangle$ is that one can decompose any vector f in this basis:

$$f = \sum_i \langle f, e_i \rangle e_i.$$

This let us simply reprove a (potentially infinite dimensional) **Pythagorean theorem**:

$$\langle f, f \rangle = \sum_i \langle f, e_i \rangle^2.$$

We also proved another version of Cauchy-Schwarz inequality:

$$\langle f, g \rangle^2 \leqslant \langle f, f \rangle \langle g, g \rangle.$$

6 Fourier series

Examples of orthonormal bases of $E = L^2([-\pi, \pi])$ with the scalar product

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f\bar{g}$$

are given by

$$\left(\frac{1}{\sqrt{2}}, \cos(m\cdot), \sin(n\cdot) \right)_{n,m \in \mathbb{N}^*} \quad \text{and} \quad \left(\frac{e^{in\cdot}}{\sqrt{2}} \right)_{n \in \mathbb{Z}}.$$

6.1 Decomposition in Fourier series

The decomposition in this basis means that for any $f \in L^2([-\pi, \pi])$, one has the following (real and complex) Fourier series of f

$$\hat{f} = \frac{a_0}{2} + \sum_{m \geqslant 1} a_m \cos(m\cdot) + b_m \sin(m\cdot)$$

and

$$\hat{f} = \sum_{n \in \mathbb{Z}} c_n e^{in\cdot}$$

were we have:

1. $a_m := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(mt) dt$,
2. $b_m := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(mt) dt$, and
3. $c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$ **Careful:** I have sometimes made the mistake of writing $1/\pi$ in c_n instead of $1/2\pi$ when solving exercises.

Note: When f is not L^2, we still define $\hat{f}$ as above when the coefficients make sense (that is essentially, when $f \in L^1$). It will however not necessarily have the property that $\|f-\hat{f}\|_{L^2}=0$.

The Pythagorean theorem in this setting is called Parseval identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|^2 = \frac{|a_0|^2}{2} + \sum_{m\geqslant 1}|a_m|^2+|b_m|^2 = \sum_{n\in\mathbb{Z}}|c_n|^2.$$

As a corollary, a function is in L^2 if and only if the above sums of squares of Fourier coefficients are finite.

6.2 Fourier series and differentiation

It turns out that whenever a function is regular enough (say C^2), one finds $\widehat{f'} = \hat{f}'$. We therefore always define the *Fourier series of the derivative* as the derivative (term by term) of the Fourier series, that is:

1. $a_m(f') := mb_m(f)$,
2. $b_m(f') := -ma_m(f)$, and
3. $c_n(f') := inc_n(f)$.

This is used to prove Poincaré-Wirtinger's inequality stating that the L^2-norm of the derivative controls the L^2-norm of the function (this is only true in compact settings!).

This moreover lets you see how regular a function is from its Fourier decomposition, namely:

$$f \in H^k \iff \sum_{m\geqslant 1} m^{2k}(|a_m|^2+|b_m|^2) < +\infty \iff \sum_{n\in\mathbb{Z}} n^{2k}|c_n|^2 < +\infty.$$

6.3 Convergence of Fourier series

A Fourier series generally does not converge to the original function in a pointwise or uniform sense.

However, from the Sobolev embedding $H^1 \subset C^0$ (only true in dimension 1!), we have the following criteria.

Theorem 6.1. *Let $\hat{f}$ as before satisfy*

$$m^2(|a_m|^2+|b_m|^2) < +\infty \iff \sum_{n\in\mathbb{Z}} n^2|c_n|^2 < +\infty,$$

then the infinite sum of functions defining $\hat{f}$ converges uniformly to a continuous C^0 function on $[-\pi,\pi]$.

If f is continuous and satisfies $f(\pi)=f(-\pi)$, then for all $t\in[-\pi,\pi]$, one has:

$$f(t) = \hat{f}(t).$$

If one has

$$m^{2(k+2)}(|a_m|^2+|b_m|^2) < +\infty \iff \sum_{n\in\mathbb{Z}} n^{2(k+1)}|c_n|^2 < +\infty,$$

then the infinite sum of functions defining $\hat{f}$ converges uniformly to a continuous C^k function (together with the derivatives) on $[-\pi,\pi]$.

Theorem 6.2. *Let $f\in C^1$ (or $f\in C^0$ and piecewise C^1) such that $f(\pi)=f(-\pi)$, then for all $t\in[-\pi,\pi]$, one has:*

$$f(t) = \hat{f}(t).$$

6.4 Fourier series and solving ODEs and PDEs

We can find linear ODEs or PDEs by finding 2π-periodic solutions that can be decomposed in Fourier series.

For ODEs: Solving

$$\sum_k p_k y^{(k)}(x) = f(x)$$

for $p_k \in \mathbb{C}$ and $\hat{f}(x) = \sum_{n\in\mathbb{Z}} f_n e^{inx}$ amounts to solving the equations:

$$\sum_k p_k (in)^k y_n = f_n$$

for every $n \in \mathbb{Z}$.

The Fourier series of the solution is then

$$\hat{y}(x) = \sum_{n\in\mathbb{Z}} y_n e^{inx}.$$

Its regularity can be read off the Fourier coefficients: $\hat{y} \in H^k$ if and only if $\sum_{n\in\mathbb{Z}} n^{2k}|y_n|^2 < +\infty$, hence $\hat{y}$ is a C^{k-1} function. In particular, one has $\sum_{n\in\mathbb{Z}} n^2|y_n|^2 < +\infty$, then $\hat{y}$ is an actual well-defined continuous function.

For PDEs: Solving

$$\sum_{k=0} p_{k,l} \partial^k_{t^k} \partial^l_{x^l} y(t,x) = f(t,x)$$

for $p_{k,l} \in \mathbb{C}$ and $\hat{f}(t,x) = \sum_{n\in\mathbb{Z}} f_n(t) e^{inx}$ amounts to solving the ODEs

$$\sum_{k,l} p_{k,l} (in)^l y_n^{(k)}(t) = f_n(t).$$

The Fourier series of the solution is then

$$\hat{y}(t,x) = \sum_{n\in\mathbb{Z}} y_n(t) e^{inx}.$$

Its regularity at t can be read off the Fourier coefficients: $\hat{y}(t,\cdot) \in H^k$ if and only if $\sum_{n\in\mathbb{Z}} n^{2k}|y_n(t)|^2 < +\infty$, hence $\hat{y}(t,\cdot)$ is a C^{k-1} function. In particular, one has $\sum_{n\in\mathbb{Z}} n^2|y_n(t)|^2 < +\infty$, then $\hat{y}(t,\cdot)$ is an actual well-defined continuous function.

6.5 Fourier series and solving ODEs and PDEs

We can find linear ODEs or PDEs by finding 2π-periodic solutions that can be decomposed in Fourier series.

For ODEs: Solving

$$\sum_k p_k y^{(k)}(x) = f(x)$$

for $p_k \in \mathbb{C}$ and $\hat{f}(x) = \sum_{n\in\mathbb{Z}} f_n e^{inx}$ amounts to solving the equations:

$$\sum_k p_k (in)^k y_n = f_n$$

for every $n \in \mathbb{Z}$.

The Fourier series of the solution is then

$$\hat{y}(x) = \sum_{n \in \mathbb{Z}} y_n e^{inx}.$$

Its regularity can be read off the Fourier coefficients: $\hat{y} \in H^k$ if and only if $\sum_{n\in\mathbb{Z}} n^{2k}|y_n|^2 < +\infty$, hence $\hat{y}$ is a C^{k-1} function. In particular, one has $\sum_{n\in\mathbb{Z}} n^2|y_n|^2 < +\infty$, then $\hat{y}$ is an actual well-defined continuous function.

For PDEs: Solving

$$\sum_{k=0} p_{k,l} \partial^k_{t^k} \partial^l_{x^l} y(t,x) = f(t,x)$$

for $p_{k,l} \in \mathbb{C}$ and $\hat{f}(t,x) = \sum_{n\in\mathbb{Z}} f_n(t) e^{inx}$ amounts to solving the ODEs

$$\sum_{k,l} p_{k,l} (in)^l y_n^{(k)}(t) = f_n(t).$$

The Fourier series of the solution is then

$$\hat{y}(t,x) = \sum_{n \in \mathbb{Z}} y_n(t) e^{inx}.$$

Its regularity at t can be read off the Fourier coefficients: $\hat{y}(t,\cdot) \in H^k$ if and only if $\sum_{n\in\mathbb{Z}} n^{2k}|y_n(t)|^2 < +\infty$, hence $\hat{y}(t,\cdot)$ is a C^{k-1} function. In particular, one has $\sum_{n\in\mathbb{Z}} n^2|y_n(t)|^2 < +\infty$, then $\hat{y}(t,\cdot)$ is an actual well-defined continuous function.

7 Systems of ODEs

One can define ODEs in any dimension (even infinite), and actually on many different geometries and settings.

7.1 First properties of first order d-dimensional ODEs

A general *d-dimensional first order ODE* is of the form:

$$Y'(t) = F(t, Y(t)),$$

where $Y(t) \in \mathbb{R}^d$, $F(t, Y(t)) \in \mathbb{R}^d$.

We will denote their coordinates: $Y(t) = (y_1(t), ..., y_d(t))$ and $F(t, Y(t)) = (F_1(t, Y(t)), ..., F_d(t, Y(t)))$.

Our existence-uniqueness proof still holds in this context (and many others!).

Theorem 7.1. *If $F : I \times \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz with respect to the $\mathbb{R}^d$-variable and continuous with respect to the I-variable, then for all $(t_0, Y_0) \in I \times \mathbb{R}$ there exists a unique solution of*

$$\begin{cases} Y'(t) = F(t, Y(t)) \\ Y(t_0) = Y_0 \end{cases}$$

that has a unique ***maximal interval of existence*** *included in I.*

If for any t, $Y \mapsto F(t, Y)$ is $L(t)$-Lipschitz, then the maximal interval of existence is I itself.

7.2 From d^{th}-order ODEs in dimension 1 to 1^{st} order ODEs in dimension d

Moreover, we can rewrite every k^{th} order ODE in dimension d as a 1^{st} order ODE in dimension $k \times d$ so the above existence-uniqueness theorem holds for any situation.

For a k^{th} order ODE in dimension 1 of the form

$$y^{(d)} = f(t, y, y', ..., y^{(d-1)}) \tag{1}$$

the process is as follows: define

$$y_k(t) := y^{(k)}(t)$$

for $0 \leqslant k \leqslant d-1$, and define $Y(t) := (y_0(t), ..., y_{d-1}(t))$.

The equation (1) rewrites $Y'(t) = F(t, Y(t))$ from the following equations coordinate by coordinate:

$$\begin{cases} y_0'(t) = y_1(t), \\ y_1'(t) = y_2(t), \\ ... \\ y_{d-1}'(t) = f(t, y_0(t), y_1(t), ..., y_{(d-1)}(t)) \end{cases}$$

The natural initial value problem specifies the values of $y(t_0), y'(t_0), ..., y^{(d-1)}(t_0)$.

7.3 First order linear ODE with constant coefficients

The simplest and most important type of ODE is linear with constant coefficients. We will see that they appear as approximations of nonlinear systems of ODEs.

They are of the form

$$Y'(t) = AY(t) \tag{2}$$

where $Y(t) \in \mathbb{R}^d$ is a vector and $A \in \mathbb{R}^{d\times d}$ is a matrix. Fixing initial conditions lets one study *one solution* at a time.

7.3.1 Fundamental matrix

On the other hand, one may want to find *all solutions* at once which can be done by finding a fundamental matrix $U(t) \in \mathbb{R}^{d\times d}$ satisfying:

$$U'(t) = AU(t) \tag{3}$$

and $\det(U(t_0)) \neq 0$ (or equivalently $\det(U(t)) \neq 0$ for any t). In this case, every column of $U(t)$ is a solution of (2). Moreover, every single solution can be obtained as $U(t)X_0$ for some $X_0 \in \mathbb{R}^d$.

$\det(U(t))$ is the *Wronskian* of the system and generalizes our second order situation.

7.3.2 Characteristic polynomial

One finds the *eigenvalues* of a matrix $A \in \mathbb{R}^{d\times d}$ and their *multiplicities* as the roots and their multiplicities of the *characteristic polynomial* of A defined as:

$$P_A(s) := \det(A - s\,\mathrm{I}_d).$$

These eigenvalues will be crucial in determining the dynamical properties of our system of ODEs.

7.4 Exponential of a matrix

We have two equivalent definitions for the *exponential of a matrix* M:

1. $e^{tM} = Y(t)$ is the unique solution to the ODE:

$$\begin{cases} Y'(t) = MY(t), \\ Y(0) = \mathrm{I}_d, \end{cases}$$

2. it is given by the sum:

$$e^{tM} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} M^k.$$

It is an important definition since any solution of $Y'(t) = AY(t)$ is equal to

$$Y(t) = e^{(t-t_0)A} Y(t_0).$$

7.4.1 Diagonalizable matrix

We have two equivalent definitions of a *diagonalizable* matrix A:

1. there exists a *basis* of eigenvectors of A,
2. there exists an invertible matrix P such that $P^{-1}AP = D$ with D diagonal: its diagonal values are the eigenvalues of A with multiplicity.

We have two simple and usual criteria implying that a matrix is diagonalizable.

Proposition 7.2. *If $A \in \mathbb{R}^{d\times d}$ has d distinct eigenvalues (or equivalently all of the eigenvalues are of multiplicity one), then it is diagonalizable.*

(this implies that with probability 1 a random matrix is diagonalizable).

Proposition 7.3. *If $A \in \mathbb{R}^{d\times d}$ is symmetric, then it is diagonalizable in an orthonormal basis and has real eigenvalues.*

Computing the exponential of a diagonalizable matrix is simple from the second definition of a diagonalizable matrix:

$$e^{tA} = Pe^{tD}P^{-1},$$

where if $D = diag(\lambda_1, ..., \lambda_d)$, then $e^{tD} = diag(e^{\lambda_1 t}, ..., e^{\lambda_d t})$.

From the first definition, we can also say that if $AV_i = \lambda_i V_i$, then $e^{tA}V_i = e^{t\lambda_i}V_i$. This is especially useful when solving ODEs: if $Y(t_0) = \sum y_i V_i$, then the solution of the ODE is given by

$$Y(t) = e^{(t-t_0)A} Y(t_0) = \sum_i e^{\lambda_i t} y_i V_i.$$

7.4.2 Nilpotent matrix

Some matrices are however not diagonalizable. The typical example being that of a *nilpotent* matrix. We have seen several definitions for them: having zero as their only eigenvalue or satisfying $N^d = 0$.

We can therefore compute their exponential which is simply a *polynomial* in the matrix!

$$e^{tN} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} N^k = \sum_{k=0}^{d} \frac{t^k}{k!} N^k.$$

7.4.3 General matrix

The classical formula for the real exponential extends to matrix exponential *when the matrices commute.*

Proposition 7.4. *Let A and B be commuting $d \times d$ matrices. Then, for all $t \in \mathbb{R}$,*

$$e^{t(A+B)} = e^{tA}e^{tB}.$$

We can actually decompose any matrix into a diagonalizable matrix and a nilpotent matrix commuting together. This lets one compute the exponential of *any* matrix.

Theorem 7.5 (Jordan, Jordan-Chevalley, Dunford, Frobenius)**.** *Let M be a $d \times d$ matrix. There exist A and N satisfying the following:*

1. $M = A + N$,
2. $AN = NA$,
3. *A is diagonalizable,*
4. *N is nilpotent.*

Consequently, we may compute the exponential as follows:

$$e^{tM} = e^{tA}e^{tN}.$$

8 (In)stability of equilibria

An *equilibrium* or *fixed point*, or *stationary point* for a system of ODEs

$$y'(t) = f(y(t))$$

is an initial value $y_0 \in \mathbb{R}^d$ satisfying $f(y_0) = 0$. By the uniqueness part of the existence-uniqueness theorem, the unique solution of $y'(t) = f(y(t))$ starting at $y(t_0) = y_0$ is the constant solution $t \mapsto y_0$ itself.

8.1 Different notions of stability

There are numerous notions of stability but we discussed the following ones.

- A *Lyapunov*-stable fixed point is such that the solutions stay close to the fixed point for all times if they start close to it.
- An *asymptotically* stable fixed point is such that any solution starting close to a fixed point converges to it as time goes to $+\infty$.
- We say that a fixed point is unstable if none of the above property is satisfied.

8.2 (In)stability of first order systems of linear ODEs

The stability of a first order linear system

$$Y'(t) = MY(t) \tag{4}$$

has the following complete characterization of the stability of the fixed point $0 \in \mathbb{R}^d$.

Theorem 8.1. *Let M be a $d \times d$-matrix as above, then 0 is*

1. *Lyapunov and asymptotically stable if and only if all of the eigenvalues of M have negative real part,*
2. *Lyapunov stable if and only if all of the eigenvalues of M have nonpositive real part and the Jordan blocks associated to eigenvalues with zero real part have no nilpotent part, and*
3. *it is unstable otherwise.*

For clarity, we will say that a linear system is **strictly stable** if M only has eigenvalues with negative real parts. We will also say that it is **strictly unstable** if it has one eigenvalue with a positive real part.

We can be a little more precise by defining the stable, unstable and center directions of the flow (we will only define it when M is *diagonalizable* here, see the lecture notes for the general situation). If M is diagonalizable, then:

- the **stable subspace** is the vector subspace spanned by the eigenvalues with negative real part
- the **center subspace** is the vector subspace spanned by the eigenvalues with zero real part, and
- the **unstable subspace** is the vector subspace spanned by the eigenvalues with positive real part.

8.3 General (in)stability of nonlinear ODEs

The *dynamical* stability of a fixed point of a nonlinear ODE can be determined by its *linear* stability, that is the stability of its linearization.

Consider the (potentially nonlinear) equation

$$y'(t) = f(y(t)) \tag{5}$$

and assume that $f(y_0) = 0$. We can then consider the linearization of $f = (f_1, ..., f_d)$ at y_0 as follows:

$$f(y) = d_{y_0} f(y - y_0) + \mathcal{O}(|y - y_0|^2),$$

where the *Jacobian* $d_{y_0} f$ is given in matrix form by

$$(d_{y_0} f)_{ij} = \partial_j f_i(y_0),$$

or

$$d_{y_0} f = \begin{bmatrix} \partial_1 f_1(y_0) & ... & \partial_d f_1(y_0) \\ ... & ... & ... \\ \partial_1 f_d(y_0) & ... & \partial_d f_d(y_0) \end{bmatrix}.$$

The stability of y_0 will be linked to its linear stability, which is the stability of the *linearized equation*:

$$y'(t) = My(t). \tag{6}$$

We have the following (almost) characterization of stable and unstable fixed points.

Theorem 8.2. *If the linearized equation (6) is* ***strictly stable*** *(that is if M only has eigenvalues with negative real part), then the nonlinear equation (5) is* ***Lyapunov and asymptotically stable****.*

If the linearized equation (6) is ***strictly unstable*** *(that is if M has at least one eigenvalue with positive real part), then the nonlinear equation (5) is* ***unstable****.*

8.4 Gradient flows

One extremely common type of ODE is the gradient flow. Most optimization algorithms used in the industry can be seen as discretized gradient flows. Indeed, gradient flows and their extensions are very natural since they are the best way to *optimize* a given function.

In more theoretical mathematics, they are also very common in geometric analysis and physics and were used to solve outstanding problems. Poincaré's conjecture, Thurston's geometrization conjecture and the generalized Smale conjecture can be solved by Ricci flow. Penrose inequality in general relativity can be solved thanks to the inverse mean curvature flow.

The *gradient flow* of a function $F : \mathbb{R}^d \to \mathbb{R}$ is given by the equation:

$$y'(t) = -\nabla F(y(t)),$$

where $\nabla F(y)$ is the gradient vector of coordinates $(\partial_1 F(y), ..., \partial_d F(y))$. The fixed points of a gradient flows are critical points of F, i.e. $y_0 \in \mathbb{R}^d$ such that $\nabla F(y_0) = 0$.

Theorem 8.3. *Let y_0 be a critical point of F.*

If it is an isolated minimum of F, then it is Lyapunov and asymptotically stable

If it is an isolated maximum or a saddle point (i.e. $\operatorname{Hess} F(y_0)$ *has positive and negative eigenvalues) then it is unstable.*

Note: In the previous notations we would have $f = \nabla F$ and the linearization of the gradient flow at a fixed point y_0 is $-\operatorname{Hess} F(y_0)$, that is the matrix of coefficients $\partial_i \partial_j F(y_0)$. It is a symmetric and real matrix if F is C^2 and therefore only has real eigenvalues and can be diagonalized in an orthonormal basis.

8.5 Hamiltonian flows

Another common type of flow, especially in physics is the *Hamiltonian flow*. Given a C^2 function $H : \mathbb{R}^{2d}$, a *Hamiltonian*, we define the Hamiltonian system by:

$$\begin{cases} x_i'(t) = \partial_{y_i} H(x(t), y(t)) \\ y_i'(t) = -\partial_{x_i} H(x(t), y(t)) \end{cases} \tag{7}$$

where we used the notation $(x_1, ...x_d, y_1, ...y_d) \in \mathbb{R}^{2d}$ or $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$.

It is the flow orthogonal to the gradient flow of H, it satisfies $\frac{d}{dt} H(x(t), y(t)) = 0$, hence stays on the level sets of H, and its fixed points are also the critical points of H. They have the following stability criterion.

Proposition 8.4. *Let (x_0, y_0) be a critical point of H. If $H(x(t), y(t)) - H(x_0, y_0)$ has a constant sign (e.g. if (x_0, y_0) is a local extremum of H), then (x_0, y_0) is a Lyapunov stable but not asymptotically stable fixed point of* (7).

The link with physics is as follows. Typically $y(t) = x'(t)$ is the **velocity** while $x(t)$ is the **position**. The Hamiltonian of common Newtonian mechanics problems decomposes as:

$$H(x, y) = W(x) + T(y)$$

where $T(y) = \frac{|y|^2}{2}$ is the *kinetic energy* (depending only on the velocity) and $W(x)$ is the *potential energy* (depending only on the position).

Newton's second law of mechanics then rewrites

$$x''(t) = -\nabla W(x(t)),$$

and by our earlier trick of going from second order to first order ODE, it is equivalent to (7). We deduced a characterization of the fixed points of this type of equation and their stability.